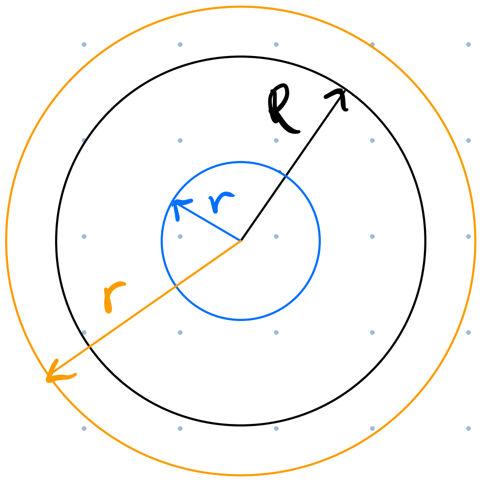
PHYS 301

# Tutorial #10
# Group Prob. Sol'ns

Nov. 29, 2024

1. Griffiths 4.10

$\vec{P} = k\vec{r}$




(a) $\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r\right)$

$= -3k$

$\sigma_b = \vec{P}\cdot\underset{\sim}{\hat{r}} = kR$

$\hat{n} = \hat{r}$ and is a position $r = R$.

(b) Since we know $\rho_b$ & $\sigma_b$, we can just apply the usual Gauss's Law.

$r < R$ (inside) Use the blue Gaussian surface.

$Q_{encl}$ is due entirely to $\rho_b$ since $\sigma_b$ is out at the surface.

$$Q_{encl} = \int \rho_b \, d\tau = -3k \frac{4}{3}\pi r^3$$
$$= -4\pi k r^3$$

$$\oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E$$

$$\therefore \cancel{4\pi} \cancel{r^2} E = -\frac{\cancel{4\pi} k r^{\cancel{3}}}{\varepsilon_0}$$

$$\boxed{\therefore \vec{E} = -\frac{k\vec{r}}{\varepsilon_0} \quad r < R}$$

$r > R$ (outside)

Use the orange Gaussian surface.

However, note that, since there is no free charge and the net bound charge must be zero, $Q_{encl} = 0 \Rightarrow \boxed{\vec{E} = 0}$

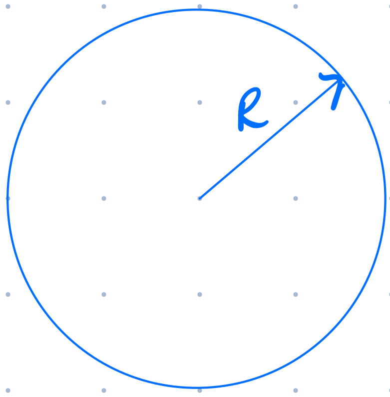
Let's confirm that $Q_b = 0$.

$$Q_b = \oint \sigma_b \, da + \int \rho_b \, d\tau$$

$$= kR \oint da - 3k \int d\tau$$

$$= kR\, 4\pi R^2 - \cancel{3}k \frac{4}{\cancel{3}} \pi R^3 = 0 \checkmark$$

2. Griffiths 4.20




$\rho_f = \rho$

Linear dielectric w/ dielectric const. $\varepsilon_r$.

Know $\oint \vec{D} \cdot d\vec{a} = Q_f$

For both $r < R$ & $r > R$, will have

$$\oint \vec{D} \cdot d\vec{a} = 4\pi r^2 D$$
$$= 4\pi r^2 \varepsilon_0 \varepsilon_r E$$

The only difference will be $\varepsilon_r = 1$ for $r > R$.

For $r < R$ $\quad Q_f = \int \rho \, d\tau = \rho \frac{4}{3}\pi r^3$

$$\therefore \cancel{4\pi} \cancel{r^2} \varepsilon_0 \varepsilon_r E = \rho \frac{\cancel{4}}{3} \cancel{\pi} r^{\cancel{3}}$$

$$\boxed{\therefore \vec{E} = \frac{\rho \vec{r}}{3\varepsilon_0 \varepsilon_r} \qquad r < R}$$

For $r > R$ $\quad Q_f = \rho \frac{4}{3}\pi R^3 \quad$ (since $\rho = 0$ for $r > R$)

$$\therefore \cancel{4\pi} r^2 \varepsilon_0 \cancelto{1}{\varepsilon_r} E = \rho \frac{\cancel{4}}{3}\cancel{\pi} R^3$$

$$\boxed{\therefore \vec{E} = \frac{\rho R^3}{3\varepsilon_0 r^2}\,\hat{r} \qquad r > R}$$

$$\Delta V = \cancelto{0}{V(\infty)} - V(0) = -\int_0^\infty \vec{E}\cdot d\vec{l}$$

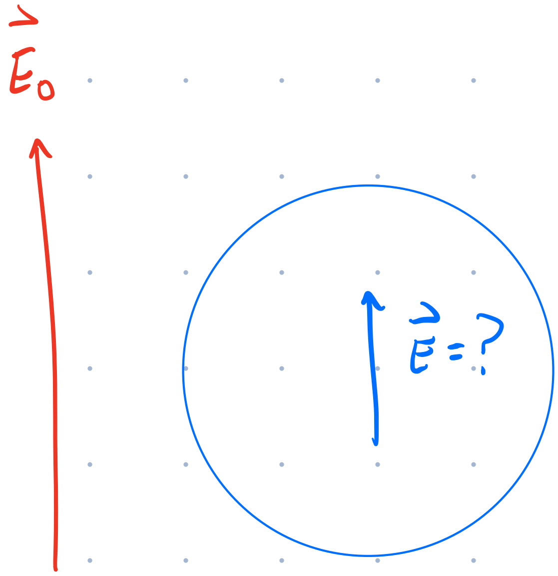
$$\therefore V(0) = \frac{\rho}{3\varepsilon_0}\left[\int_0^R \frac{r}{\varepsilon_r}\,dr + \int_R^\infty \frac{R^3}{r^2}\,dr\right]$$

$$= \frac{\rho}{3\varepsilon_0}\left[\frac{R^2}{2\varepsilon_r} - \frac{R^3}{r}\Big|_R^\infty\right]$$

$$\therefore V(0) = \frac{\rho}{3\varepsilon_0}\left[\frac{R^2}{2\varepsilon_r} + R^2\right] = \frac{\rho R^2}{3\varepsilon_0}\left[\frac{1}{2\varepsilon_r} + 1\right]$$

or $$\boxed{V(0) = \frac{\rho R^2}{6\varepsilon_0\varepsilon_r}\left[1 + 2\varepsilon_r\right]}$$

3. Griffiths 4.23




linear dielectric $\varepsilon_r$

Goal: Show that inside the dielectric $\vec{E} = \dfrac{3}{\varepsilon_r + 2} \vec{E}_0$

(i) Assume that $\vec{E} = \vec{E}_0$

$\rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$ but $\varepsilon_r = 1 + \chi_e$

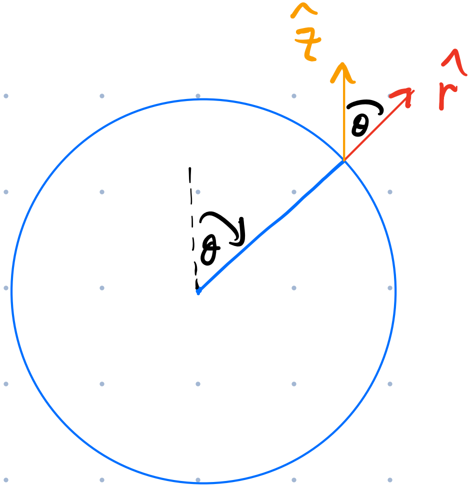
$= \varepsilon_0 (\varepsilon_r - 1) \vec{E}$ $\quad \therefore \chi_e = \varepsilon_r - 1$

This polarization establishes bound charges

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0 \text{ since } \vec{P} \text{ is uniform}$$

$$\sigma_b = \vec{P} \cdot \hat{r} = \varepsilon_0 (\varepsilon_r - 1) E_0 \, \hat{z} \cdot \hat{r}$$

$$= \varepsilon_0 (\varepsilon_r - 1) E_0 \cos\theta$$




Now, we previously found (using separation of variables) that for a sphere:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

for $r<R$, $B_l = 0$ since $\frac{1}{r^{l+1}}$ diverges @ $r=0$

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

for $r>R$, $A_l = 0$ since $r^l$ diverges for $r \to \infty$

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Require $V_{in} = V_{out}$ @ $r=R$ (i)

& $E_{out}^{\perp} - E_{in}^{\perp} = \frac{\sigma}{\varepsilon_0}$ @ $r=R$

$$\therefore \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = -\frac{\sigma}{\varepsilon_0} \quad \text{(ii)}$$

Apply (ii) $$\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta)$$

$$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = \sum_{l=0}^{\infty} -(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta)$$

$$\therefore \sum_{l=0}^{\infty}\left[\frac{(l+1)B_l}{R^{l+2}} + l A_l R^{l-1}\right] P_l(\cos\theta)$$

$$= \frac{\cancel{\varepsilon_0}(\varepsilon_r - 1)E_0 \cos\theta}{\cancel{\varepsilon_0}}$$

must have only the
$l=1$ term w/ $P_1(\cos\theta) = \cos\theta$

$$\frac{2B}{R^3} + A = (\varepsilon_r - 1)E_0 \qquad (ii')$$

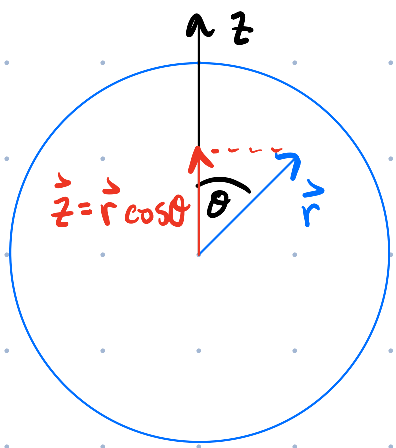
Next, apply (i) w/ $l=1$

$$AR = \frac{B}{R^2} \quad \therefore \ B = AR^3 \quad \text{sub into } (ii')$$

$$\frac{2A\cancel{R^3}}{\cancel{R^3}} + A = (1+\varepsilon_r)E_0$$

$$\therefore \ \boxed{A = \frac{(\varepsilon_r - 1)}{3}E_0 \qquad B = \frac{(\varepsilon_r-1)R^3}{3}E_0}$$

So $V_{in} = \frac{1}{3}(\varepsilon_r - 1)E_0 \underbrace{r\cos\theta}_{z}$




$$\therefore \ V_{in} = \frac{1}{3}(\varepsilon_r - 1)E_0 z$$

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{\partial}{\partial z} V_{in}\,\hat{z}$$

$$\therefore \vec{E}_{in} = \frac{-1}{3}(\varepsilon_r - 1)E_0\,\hat{z}$$

Summary:

| $\vec{E}$ | $\vec{P}$ | $\sigma_b$ |
|---|---|---|
| $\vec{E}_0$ | $\vec{P}_0 = \varepsilon_0(\varepsilon_r-1)\vec{E}_0$ | $\sigma_0 = \varepsilon_0(\varepsilon_r-1)E_0\cos\theta$ |
| $\vec{E}_1 = -\frac{1}{3}(\varepsilon_r-1)\vec{E}_0$ | $\vec{P}_1 = \frac{-\varepsilon_0(\varepsilon_r-1)^2}{3}\vec{E}_0$ | $\sigma_1 = -\frac{\varepsilon_0(\varepsilon_r-1)^2}{3}E_0\cos\theta$ |
| $\vec{E}_2 = \frac{(\varepsilon_r-1)^2}{9}\vec{E}_0$ | $\vec{P}_2 = \frac{\varepsilon_0(\varepsilon_r-1)^3}{9}\vec{E}_0$ | $\sigma_2 = \frac{\varepsilon_0(\varepsilon_r-1)^3}{9}E_0\cos\theta$ |
| $\vec{E}_3 = -\frac{(\varepsilon_r-1)^3}{27}\vec{E}_0$ | $\vec{P}_3 = -\frac{\varepsilon_0(\varepsilon_r-1)^4}{27}\vec{E}_0$ | $\sigma_3 = \frac{-\varepsilon_0(\varepsilon_r-1)^4}{27}E_0\cos\theta$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\vec{E}_N = \frac{(\varepsilon_r-1)^N}{(-3)^N}\vec{E}_0$ | $\vec{P}_N = \varepsilon_0\frac{(\varepsilon_r-1)^{N+1}}{(-3)^N}\vec{E}_0$ | $\sigma_N = \frac{\varepsilon_0(\varepsilon_r-1)^{N+1}}{(-3)^N}E_0\cos\theta$ |

Process is as follows:

bound charge

- $\vec{E}_0$ establishes polarization $\vec{P}_0$ & $\sigma_0$
- $\vec{P}_0 / \sigma_0$ creates its own $\vec{E}_1$ $\therefore \vec{E} \to \vec{E}_0 + \vec{E}_1$
- $\vec{E}_1$ makes its own contribution to the polarization ($\vec{P}_1$ & $\sigma_1$).
- $\vec{P}_1 / \sigma_1$ creates its own $\vec{E}_2$ $\therefore \vec{E} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2$

$$\vdots$$

$$\vec{E}_{net} = \sum_{i=0}^{\infty} \vec{E}_i$$

Likewise $\vec{P}_{net} = \sum_{i=0}^{\infty} \vec{P}_i$ , $\sigma_{net} = \sum_{i=0}^{\infty} \sigma_i$

$$\therefore \vec{E}_{in} = \underbrace{\sum_{i=0}^{\infty} \left( \frac{\varepsilon_r - 1}{-3} \right)^i}_{\text{Geometric series}} \vec{E}_0$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\therefore \vec{E}_{in} = \frac{1}{1 + \frac{\varepsilon_r - 1}{3}} \vec{E}_0 = \frac{1}{\frac{3 + \varepsilon_r - 1}{3}} \vec{E}_0$$

$$\Rightarrow \boxed{\vec{E}_{in} = \left(\frac{3}{\varepsilon_r + 2}\right) \vec{E}_0}$$

What about $\vec{P}$?

$$\vec{P} = \varepsilon_0 (\varepsilon_r - 1) \vec{E}_{in}$$

$$\boxed{\therefore \vec{P} = \frac{3\varepsilon_0 (\varepsilon_r - 1)}{\varepsilon_r + 2} \vec{E}_0}$$

Should be consistent w/

$$\vec{P} = \sum_{i=0}^{\infty} \vec{P}_i = \varepsilon_0 \sum_{i=0}^{\infty} \frac{(\varepsilon_r - 1)^{i+1}}{(-3)^i} \vec{E}_0$$

$$= \varepsilon_0 (\varepsilon_r - 1) \vec{E}_0 \sum_{i=0}^{\infty} \left(\frac{\varepsilon_r - 1}{-3}\right)^i$$

same.

$$\therefore \vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E}_0 \underbrace{\frac{1}{1+\frac{\varepsilon_r - 1}{3}}}_{\frac{3}{\varepsilon_r+2}}$$

$$\boxed{\therefore \vec{P} = \frac{3\varepsilon_0(\varepsilon_r - 1)}{(\varepsilon_r + 2)}\vec{E}_0}$$

What about $\sigma_b$?

$$\boxed{\sigma_b = \vec{P}\cdot\hat{r} = \frac{3\varepsilon_0(\varepsilon_r-1)}{(\varepsilon_r+2)} E_0 \underbrace{\cos\theta}_{\hat{z}\cdot\hat{r}}}$$

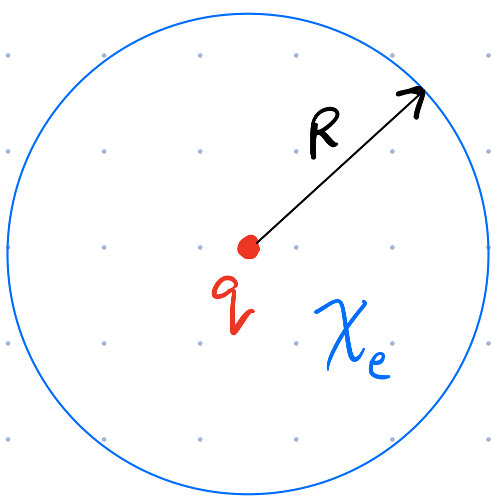
same.

Should be consistent w/

$$\sigma_b = \sum_{i=0}^{\infty} \sigma_i = \varepsilon_0(\varepsilon_r - 1) E_0 \cos\theta \underbrace{\sum_{i=0}^{\infty}\left(\frac{\varepsilon_r - 1}{-3}\right)^i}_{\frac{3}{\varepsilon_r+2}}$$

$$\boxed{\therefore\ \sigma_b = \frac{3\varepsilon_0(\varepsilon_r - 1)}{(\varepsilon_r + 2)} E_0 \cos\theta}$$

4. Griffiths 4.35




$$\oint \vec{D} \cdot d\vec{a} = \underbrace{Q_f}_{q}$$

$$\therefore D\,4\pi r^2 = q$$ valid for $\forall\, r$.

$$\varepsilon_0 E + P = \frac{q}{4\pi r^2}$$

$$\varepsilon_0 (1+\chi_e) E = \frac{q}{4\pi r^2}$$

$r < R$: $$\boxed{\vec{E} = \frac{q}{4\pi \varepsilon_0 \varepsilon_r r^2}\,\hat{r}}$$

$r > R$: $\chi_e = 0$ $\therefore$ $$\boxed{\vec{E} = \frac{q}{4\pi \varepsilon_0 r^2}\,\hat{r}}$$ $\vec{E}$ due to just the pt. charge.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$\uparrow$ $\vec{E}$ inside the sphere.

$$\boxed{\therefore \vec{P} = \frac{\chi_e q}{4\pi \varepsilon_r r^2}\,\hat{r}}$$

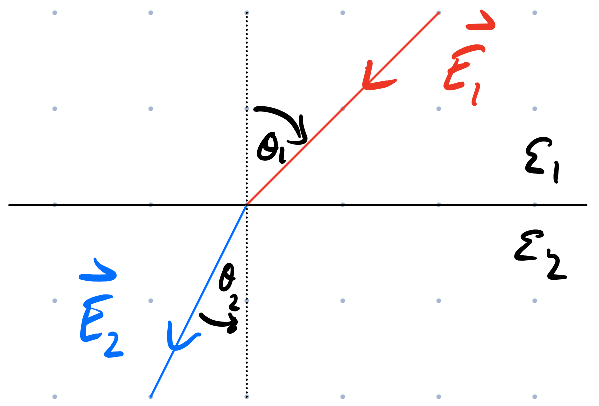
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\chi_e q}{\varepsilon_r}\frac{1}{4\pi}\underbrace{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)}_{\delta^3(\vec{r})}$$

$$\boxed{\therefore \rho_b = -\frac{\chi_e q}{\varepsilon_r}\delta^3(\vec{r})}$$

negative bound charge @ centre of dielectric

$$\boxed{\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = \frac{\chi_e q}{4\pi\varepsilon_r R^2}}$$

5. Griffith's 4.36




Apply the boundary conditions for the electric displacement:

(i) $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

(ii) $D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$

(i) $\sigma_f = 0 \quad \therefore \quad D^{\perp}_{above} = D^{\perp}_{below}$

$$\Downarrow$$

$$\varepsilon_1 E^{\perp}_{above} = \varepsilon_2 E^{\perp}_{below}$$

$$\Downarrow$$

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \quad a$$

(ii) $\varepsilon_1 E^{\parallel}_{above} - \varepsilon_2 E^{\parallel}_{below} = \varepsilon_0 \chi_1 E^{\parallel}_{above} - \varepsilon_0 \chi_2 E^{\parallel}_{below}$

$$\varepsilon_1 E_1 \sin\theta_1 - \varepsilon_2 E_2 \sin\theta_2 = \varepsilon_0 \chi_1 E_1 \sin\theta_1 - \varepsilon_0 \chi_2 E_2 \sin\theta_2$$

$$\therefore \cancel{\varepsilon_0}\underbrace{(\varepsilon_{r_1} - \chi_1)}_{\substack{1+\chi_1-\chi_1 \\ =1}} E_1 \sin\theta_1 = \cancel{\varepsilon_0}\underbrace{(\varepsilon_{r_2} - \chi_2)}_{\substack{1+\chi_2-\chi_2 \\ =1}} E_2 \sin\theta_2$$

$$\therefore E_1 \sin\theta_1 = E_2 \sin\theta_2 \qquad (b)$$

$$\frac{(b)}{(a)} = \frac{\cancel{E_1} \sin\theta_1}{\varepsilon_1 \cancel{E_1} \cos\theta_1} = \frac{\cancel{E_2} \sin\theta_2}{\varepsilon_2 \cancel{E_2} \cos\theta_2}$$

$$\therefore \frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2} \Rightarrow \boxed{\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_1}{\varepsilon_2}}$$

or $\quad \varepsilon_2 \tan\theta_2 = \varepsilon_1 \tan\theta_1$